PROCEEDINGS OF THE ROYAL SOCIETY A

Proc. R. Soc. A (2012) **468**, 3685–3704
doi:10.1098/rspa.2012.0089
Published online 27 June 2012

Existence of affine realizations for Lévy term structure models

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We investigate the existence of affine realizations for term structure models driven by Lévy processes. It turns out that we obtain more severe restrictions on the volatility than in the classical diffusion case without jumps. As special cases, we study constant direction volatilities and the existence of short-rate realizations.

Keywords: Lévy term structure model; invariant foliation; affine realization; short-rate realization

1. Introduction

A zero coupon bond with maturity T is a financial asset that pays the holder one unit of cash at T. Its price at $t \le T$ can be written as the continuous discounting of one unit of cash,

$$P(t, T) = \exp\left(-\int_t^T f(t, s)\,\mathrm{d}s\right),$$

where $f(t, T)$ is the rate prevailing at time t for instantaneous borrowing at time T, also called the forward rate for date T. The classical continuous framework for the evolution of the forward rates goes back to Heath, Jarrow and Morton (HJM Heath *et al.* 1992). They assume that, for every date T, the forward rates $f(t, T)$ follow an Itô process of the form

$$\mathrm{d}f(t, T) = \alpha_{\mathrm{HJM}}(t, T)\,\mathrm{d}t + \sigma(t, T)\,\mathrm{d}W_t, \quad t \in [0, T], \tag{1.1}$$

where W is a Wiener process.

In this study, we consider Lévy term structure models that generalize the classical HJM framework by replacing the Wiener process W in (1.1) by a more general Lévy process X, also taking into account the occurrence of jumps. This extension has been proposed by Eberlein & Raible (1999), Eberlein & Özkan (2003), Eberlein *et al.* (2005) and Eberlein & Kluge (2006*a*,*b*, 2007). Other approaches in order to generalize the classical HJM framework can be found in Björk *et al.* (1997*a*,*b*), Carmona & Tehranchi (2006) and, for example, in Shirakawa (1991), Jarrow & Madan (1995) and Hyll (2000).

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Received 10 February 2012
Accepted 1 June 2012

In the following, we therefore assume that, for every date T, the forward rates $f(t, T)$ follow an Itô process,

$$\mathrm{d}f(t, T) = \alpha_{\mathrm{HJM}}(t, T)\,\mathrm{d}t + \sigma(t, T)\,\mathrm{d}X_t, \quad t \in [0, T],$$

with X being a Lévy process. Note that such an HJM interest rate model is an infinite-dimensional object because for every date of maturity $T \geq 0$, we have an Itô process.

There are several reasons why, in practice, we are interested in the existence of a finite-dimensional realization, that is, the forward rate evolution being described by a finite-dimensional state process. Such a finite-dimensional realization ensures larger analytical tractability of the model, for example, in view of option pricing (Duffie & Kan 1996). Moreover, as argued in Baudoin & Teichmann (2005), HJM models without a finite-dimensional realization do not seem reasonable because then the support of the forward rate curves $f(t, t+\cdot)$, $t > 0$, becomes too large, and hence any 'shape' of forward rate curves, which we assume from the beginning to model the market phenomena, is destroyed with positive probability.

For classical HJM models driven by a Wiener process, the construction of finite-dimensional realizations for particular volatility structures has been treated in Jeffrey (1995), Ritchken & Sankarasubramanian (1995), Duffie & Kan (1996), Bhar & Chiarella (1997), Inui & Kijima (1998), Björk & Christensen (1999), Björk & Gombani (1999) and Chiarella & Kwon (2001, 2003), and finally, the problem concerning the existence of finite-dimensional realizations has completely been solved in Björk & Svensson (2001), Björk & Landén (2002), Filipović & Teichmann (2003), see also Filipović & Teichmann (2004) and Tappe (2010). A survey about the topic can be found in Björk (2003).

However, there are only very few references, such as Eberlein & Raible (1999), Küchler & Naumann (2003), Gapeev & Küchler (2006) and Hyll (2000), that deal with affine realizations for term structure models with jumps.

The purpose of this study is to investigate when a Lévy-driven term structure model admits an affine realization.

The main idea is to switch to the Musiela parametrization of forward curves $r_t(x) = f(t, t+x)$ (Musiela 1993), and to consider the forward rates as the solution of a stochastic partial differential equation (SPDE), the so-called Heath–Jarrow–Morton–Musiela (HJMM) equation,

$$\left.\begin{aligned} &\mathrm{d}r_t = \left(\frac{\mathrm{d}}{\mathrm{d}x} r_t + \alpha_{\mathrm{HJM}}(r_t)\right)\mathrm{d}t + \sigma(r_{t-})\,\mathrm{d}X_t \\ \text{and}\quad &r_0 = h_0, \end{aligned}\right\} \tag{1.2}$$

on a suitable Hilbert space H of forward curves, where $\mathrm{d}/\mathrm{d}x$ denotes the differential operator that is generated by the strongly continuous semigroup $(S_t)_{t \geq 0}$ of shifts. Such models have been investigated in Peszat & Zabczyk (2007*b*), Filipović & Tappe (2008) and Marinelli (2010).

The bank account B is the riskless asset that starts with one unit of cash and grows continuously at time t with the short rate $r_t(0)$, i.e.

$$B(t) = \exp\left(\int_0^t r_s(0)\,\mathrm{d}s\right), \quad t \geq 0.$$

According to Delbaen & Schachermayer (1994), the implied bond market that we can now express as

$$P(t, T) = \exp\left(-\int_0^{T-t} r_t(x)\,\mathrm{d}x\right), \quad 0 \le t \le T,$$

is free of arbitrage if there exists an equivalent (local) martingale measure $\mathbb{Q} \sim \mathbb{P}$ such that the discounted bond prices

$$\frac{P(t, T)}{B(t)}, \quad t \in [0, T],$$

are local $\mathbb{Q}$-martingales for all maturities T. If we formulate the HJMM equation (1.2) with respect to such an equivalent martingale measure $\mathbb{Q} \sim \mathbb{P}$, then the drift is determined by the volatility, i.e. $\alpha_{\mathrm{HJM}} : H \to H$ in (1.2) is given by the HJM drift condition

$$\alpha_{\mathrm{HJM}}(h) = \frac{\mathrm{d}}{\mathrm{dx}}\Psi\left(-\int_0^{\bullet} \sigma(h)(\eta)\,\mathrm{d}\eta\right) = -\sigma(h)\Psi'\left(-\int_0^{\bullet} \sigma(h)(\eta)\,\mathrm{d}\eta\right), \quad h \in H, \tag{1.3}$$

where Ψ denotes the cumulant generating function of the Lévy process (Eberlein & Özkan 2003, §2.1).

As in Björk & Svensson (2001), Björk & Landén (2002) and Filipović & Teichmann (2003), we can now regard the problem from a geometric point of view, i.e. the forward rate process has to stay on a collection of finite-dimensional affine manifolds indexed by the time t, a so-called foliation.

In general, invariance of a manifold for a stochastic process with jumps is a difficult issue because we have to ensure that the process does not jump out of the manifold. This problem has been addressed in Kurtz *et al.* (1995), where the authors consider a particular Stratonovich type integral (introduced by Markus (1978, 1981)) which, intuitively speaking, ensures that the jumps of a stochastic differential equation with vector fields being tangential to a given manifold $\mathcal{M}$, map the manifold $\mathcal{M}$ onto itself.

In this study, we avoid this problem by focusing on affine realizations because for affine manifolds, the jumps will automatically be captured, provided the volatility $h \mapsto \sigma(h)$ is tangential at each point of the manifold. Hence, in our framework, the stochastic integral in (1.2) is the usual Itô integral.

Although the jumps of the Lévy process X do not cause problems in this respect, that is, we do not have to worry that the solution r jumps out of the manifold, our investigations will show—and this is due to the particular structure of the HJM drift term α_{HJM} in (1.3) which ensures the absence of arbitrage—that we obtain more severe restrictions on the volatility σ than in the classical diffusion case.

The remainder of this text is organized as follows. In §2, we provide results on invariant foliations and on affine realizations for SPDEs driven by Lévy processes. Then, we introduce the term structure model in §3. After these preparations, in §§4 and 5, we present necessary and sufficient conditions for the existence of affine realizations for Lévy term structure models. In §6, we study constant volatilities, and in §7 constant direction volatilities and consequences for the existence of short-rate realizations. For the sake of lucidity, appendix A provides some auxiliary results that are needed in this text.

2. Invariant foliations for stochastic partial differential equations driven by Lévy processes

In this section, we provide results on invariant foliations for SPDEs driven by Lévy processes that we will apply to the HJMM equation (1.2) later on. The proofs of our results are similar to those from Tappe (2010, §§2,3), where analogous statements for Wiener-driven SPDEs are provided. Indeed, owing to the affine structure of a foliation, the Lévy process cannot jump out of the foliation. We refer the reader to Tappe (2010, §§2,3) for more details and explanations about invariant foliations.

From now on, let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions, and let X be a real-valued, square-integrable Lévy process with Gaussian part $c \geq 0$ and Lévy measure F. In order to avoid trivialities, we assume that $c + F(\mathbb{R}) > 0$. Here, we shall deal with SPDEs of the type

$$\left.\begin{aligned} \mathrm{d}r_t &= (Ar_t + \alpha(r_t))\,\mathrm{d}t + \sigma(r_{t-})\,\mathrm{d}X_t \\ \text{and} \qquad r_0 &= h_0, \end{aligned}\right\} \tag{2.1}$$

on a separable Hilbert space H. In (2.1), the operator $A : \mathcal{D}(A) \subset H \to H$ is the infinitesimal generator of a C_0-semigroup $(S_t)_{t\geq 0}$ on H, and $\alpha, \sigma : H \to H$ are measurable mappings. We refer to Peszat & Zabczyk (2007*a*) for general information about SPDEs driven by Lévy processes.

In the following, let $V \subset H$ be a finite-dimensional linear subspace.

Definition 2.1. A family $(\mathcal{M}_t)_{t\geq 0}$ of affine subspaces $\mathcal{M}_t \subset H$, $t \geq 0$, is called a foliation generated by V if there exists $\psi \in C^1(\mathbb{R}_+; H)$ such that

$$\mathcal{M}_t = \psi(t) + V, \quad t \geq 0.$$

In what follows, let $(\mathcal{M}_t)_{t\geq 0}$ be a foliation generated by the subspace V.

Definition 2.2. The foliation $(\mathcal{M}_t)_{t\geq 0}$ is called invariant for (2.1) if for every $t_0 \in \mathbb{R}_+$ and $h \in \mathcal{M}_{t_0}$, there exists a weak solution $(r_t)_{t\geq 0}$ for (2.1) with $r_0 = h$ having càdlàg sample paths such that

$$\mathbb{P}(r_t \in \mathcal{M}_{t_0+t}) = 1, \quad \text{for all } t \geq 0.$$

Definition 2.2 of an invariant foliation slightly deviates from that in Tappe (2010), as it includes the existence of a weak solution for (2.1). However, the proofs of the following results are similar to that in Tappe (2010).

Theorem 2.3. *We suppose that the following conditions are satisfied*:

— *the foliation $(\mathcal{M}_t)_{t\geq 0}$ is invariant for (2.1) and*
— *the mappings α and σ are continuous.*

Then, for all $t \geq 0$, *the following conditions hold true*:

$$\mathcal{M}_t \subset \mathcal{D}(A), \tag{2.2}$$

$$\nu(h) \in T\mathcal{M}_t, \quad h \in \mathcal{M}_t, \tag{2.3}$$

and

$$\sigma(h) \in V, \quad h \in \mathcal{M}_t. \tag{2.4}$$

In theorem 2.3, the mapping $\nu : \mathcal{D}(A) \to H$ is defined by $\nu := A + \alpha$, and $T\mathcal{M}_t$ denotes the tangent space of the foliation at time t (Tappe 2010).

Theorem 2.4. *We suppose that the following conditions are satisfied*:

— *conditions* (*2.2*)–(*2.4*) *hold true and*
— α *and* σ *are Lipschitz continuous.*

Then, the foliation $(\mathcal{M}_t)_{t\geq 0}$ *is invariant for* (*2.1*).

The previous results lead to the following definition of an affine realization.

Definition 2.5. Let $V \subset H$ be a finite-dimensional subspace.

— The SPDE (2.1) has an *affine realization generated by* V, if for each $h_0 \in \mathcal{D}(A)$, there exists a foliation $(\mathcal{M}_t^{(h_0)})_{t\geq 0}$ generated by V with $h_0 \in \mathcal{M}_0^{(h_0)}$, which is invariant for (2.1).
— In this case, we call $d := \dim V$ the *dimension* of the affine realization.
— The SPDE (2.1) has an affine realization, if it has an affine realization generated by some subspace V.
— An affine realization generated by some subspace V is called *minimal*, if for another affine realization generated by some subspace W, we have $V \subset W$.

Lemma 2.6. *Let* $V \subset H$ *be a finite-dimensional subspace. We suppose that the following conditions are satisfied*:

— *the SPDE* (*2.1*) *has an affine realization generated by* V *and*
— α *and* σ *are continuous.*

Then, we have $\sigma(h) \in V$, *for all* $h \in H$.

Proof. Using theorem 2.3, we have $\sigma(h) \in V$ for all $h \in \mathcal{D}(A)$. Because σ is continuous, $\mathcal{D}(A)$ is dense in H, and V is closed, we deduce that $\sigma(h) \in V$ for all $h \in H$. ■

3. Presentation of the term structure model

In this section, we shall introduce the Lévy term structure model. Recall that $c \geq 0$ denotes the Gaussian part and F the Lévy measure of the Lévy process X.

We define the domain

$$\mathcal{D}(\Psi) := \left\{ z \in \mathbb{R} : \int_{\{|x|>1\}} \mathrm{e}^{zx} F(\mathrm{d}x) < \infty \right\}$$

and the cumulant generating function

$$\Psi : \mathcal{D}(\Psi) \to \mathbb{R}, \quad \Psi(z) := bz + \frac{c}{2} z^2 + \int_{\mathbb{R}} (\mathrm{e}^{zx} - 1 - zx) F(\mathrm{d}x),$$

where $b \in \mathbb{R}$ denotes the drift of X. Note that Ψ is of class C^∞ in the interior of $\mathcal{D}(\Psi)$. In what follows, we assume that $K \subset \mathcal{D}(\Psi)$ for some compact interval K with $0 \in \operatorname{Int} K$. Then, the cumulant-generating function Ψ is even analytic on the interior of K, and thus, for some $\epsilon > 0$, we obtain the power series representation

$$\Psi(z) = \sum_{n=0}^{\infty} a_n z^n, \quad z \in (-\epsilon, \epsilon), \tag{3.1}$$

where the coefficients (a_n) are given by

$$a_n = \frac{\Psi^{(n)}(0)}{n!}, \quad n \in \mathbb{N}_0.$$

Note that

$$a_2 = \frac{1}{2} \left(c + \int_{\mathbb{R}} x^2 F(\mathrm{d}x) \right) \quad \text{and} \quad a_n = \frac{1}{n!} \int_{\mathbb{R}} x^n F(\mathrm{d}x), \quad \text{for } n \geq 3. \tag{3.2}$$

We fix an arbitrary constant $\beta > 0$ and denote by H_β the space of all absolutely continuous functions $h : \mathbb{R}_+ \to \mathbb{R}$ such that

$$\|h\|_\beta := \left(|h(0)|^2 + \int_{\mathbb{R}_+} |h'(x)|^2 \mathrm{e}^{\beta x} \,\mathrm{d}x \right)^{1/2} < \infty. \tag{3.3}$$

Spaces of this kind have been introduced in Filipović (2001). We also refer to Tappe (2010, §4), where some relevant properties have been summarized. Let H_β^0 be the subspace

$$H_\beta^0 := \left\{ h \in H_\beta : \lim_{x \to \infty} h(x) = 0 \right\}. \tag{3.4}$$

We fix arbitrary constants $0 < \beta < \beta'$.

Definition 3.1. Let $H_{\beta,\beta'}^{\Psi}$ be the set of all mappings $\sigma : H_\beta \to H_{\beta'}^0$ such that

$$-\int_0^x \sigma(h)(\eta) \,\mathrm{d}\eta \in K, \quad \text{for all } h \in H_\beta \text{ and } x \in \mathbb{R}_+.$$

For a volatility $\sigma \in H_{\beta,\beta'}^{\Psi}$, we define the drift α_{HJM} according to the HJM drift condition (1.3).

Remark 3.2. Owing to lemma 2.6, throughout this text, we will deal with volatility structures of the form

$$\sigma(h) = \sum_{i=1}^{p} \Phi_i(h)\lambda_i, \quad h \in H_\beta, \tag{3.5}$$

with real-valued mappings $\Phi_1, \ldots, \Phi_p : H_\beta \to \mathbb{R}$ and functions $\lambda_1, \ldots, \lambda_p \in H^0_{\beta'}$. By Tappe (2010, lemma 4.3), we have $\Lambda_1, \ldots, \Lambda_p \in H_\beta$, where $\Lambda_j := \int_0^\bullet \lambda(\eta)\mathrm{d}\eta$ for $j = 1, \ldots, p$, and hence, these functions are bounded. Thus, any volatility σ of the form (3.5), for which the mappings $\Phi_1, \ldots, \Phi_p$ are suitably bounded, belongs to $H^{\Psi}_{\beta,\beta'}$.

We denote by $(S_t)_{t\geq 0}$ the shift-semigroup on H_β. From the theory of strongly continuous semigroups (Pazy 1983), it is well known that the domain $\mathcal{D}(\mathrm{d}/\mathrm{d}x)$, endowed with the graph norm

$$\|h\|_{\mathcal{D}(\mathrm{d}/\mathrm{d}x)} := \left(\|h\|^2_\beta + \left\| \left(\frac{\mathrm{d}}{\mathrm{d}x}\right) h \right\|^2_\beta \right)^{1/2}, \quad h \in H_\beta,$$

itself is a separable Hilbert space, and that $(S_t)_{t\geq 0}$ is also a C_0-semigroup on $(\mathcal{D}(\mathrm{d}/\mathrm{d}x), \|\cdot\|_{\mathcal{D}(\mathrm{d}/\mathrm{d}x)})$. Using similar techniques as in Tappe (2010, §4) and Filipović & Tappe (2008, §4), we obtain the following auxiliary result.

Lemma 3.3. *Let $\sigma \in H^{\Psi}_{\beta,\beta'}$ be arbitrary.*

- *We have $\alpha_{\mathrm{HJM}}(h) \in H^0_\beta$, for all $h \in H_\beta$.*
- *If σ is continuous, then α_{HJM} is continuous, too.*
- *If σ is Lipschitz continuous and bounded, then α_{HJM} is Lipschitz continuous.*
- *If $\sigma(\mathcal{D}(\mathrm{d}/\mathrm{d}x)) \subset \mathcal{D}(\mathrm{d}/\mathrm{d}x)$ and σ is Lipschitz continuous and bounded on $(\mathcal{D}(\mathrm{d}/\mathrm{d}x), \|\cdot\|_{\mathcal{D}(\mathrm{d}/\mathrm{d}x)})$, then $\alpha_{\mathrm{HJM}}(\mathcal{D}(\mathrm{d}/\mathrm{d}x)) \subset \mathcal{D}(\mathrm{d}/\mathrm{d}x)$ and α_{HJM} is Lipschitz continuous on $(\mathcal{D}(\mathrm{d}/\mathrm{d}x), \|\cdot\|_{\mathcal{D}(\mathrm{d}/\mathrm{d}x)})$.*

Note that the HJMM equation (1.2) is a particular example of the SPDE (2.1) on the state space $H = H_\beta$ with infinitesimal generator $A = \mathrm{d}/\mathrm{d}x$ and $\alpha = \alpha_{\mathrm{HJM}}$. Owing to lemma 3.3, we can apply all previous results about invariant foliations from §2 in the following.

4. Necessary conditions for the existence of affine realizations

In this section, we shall derive necessary conditions for the existence of affine realizations for Lévy term structure models.

Throughout this section, we assume that the HJMM equation (1.2) has an affine realization generated by some subspace $V \subset H_\beta$. We suppose that the volatility $\sigma \in H^{\Psi}_{\beta,\beta'}$ is continuous. According to lemma 3.3, the drift α_{HJM} is continuous, too. Recall that F denotes the Lévy measure of the driving Lévy process X in (1.2). We suppose there exists an index $n_0 \in \mathbb{N}$ such that

$$\int_{\mathbb{R}} x^n F(\mathrm{d}x) \neq 0, \quad \text{for all } n \geq n_0, \tag{4.1}$$

and we suppose that for each $\lambda \in V$ with $\lambda \neq 0$, we have

$$\lambda|_{[0,\kappa]} \not\equiv 0, \quad \text{for all } \kappa > 0. \tag{4.2}$$

We fix an arbitrary $h_0 \in \mathcal{D}(\mathrm{d}/\mathrm{d}x)$ and define the linear space $W := \langle \sigma(h_0 + V) \rangle$. Recall that a function $v \in \mathcal{D}((\mathrm{d}/\mathrm{d}x)^\infty)$ is called *quasi-exponential* if

$$\dim \left\langle \left(\frac{\mathrm{d}}{\mathrm{d}x} \right)^n v : n \in \mathbb{N}_0 \right\rangle < \infty.$$

Theorem 4.1. *The following statements are true*:

- *we have* $\sigma(H_\beta) \subset V$;
- *for every subspace* $U \subset W$ *with* $\dim U \geq 1$ *and each set* $Y \subset \sigma(h_0 + V)$ *with* $Y \cap U \neq \emptyset$, *the set* $Y \cap U$ *cannot be open in* U; *and*
- *if* σ *is constant on* $h_0 + V$, *then each* $v \in V$ *is quasi-exponential, and we have* $\langle (\mathrm{d}/\mathrm{d}x)^n v : n \in \mathbb{N}_0 \rangle \subset V$.

Remark 4.2. The relation $\sigma(H_\beta) \subset V$ implies that the volatility σ is of the form

$$\sigma(h) = \sum_{i=1}^{p} \Phi_i(h) \lambda_i, \quad h \in H_\beta,$$

with real-valued mappings $\Phi_1, \ldots, \Phi_p : H_\beta \to \mathbb{R}$ and functions $\lambda_1, \ldots, \lambda_p \in H^0_{\beta'}$. Theorem 4.1 shows that we obtain restrictions on the mappings $\Phi_1, \ldots, \Phi_p$, which mean that their range cannot be arbitrarily rich. Such restrictions do not occur in the Wiener driven case (Björk & Svensson 2001; Björk & Landén 2002; Filipović & Teichmann 2003, 2004; Tappe 2010).

Before we start with the proof of theorem 4.1, we shall derive some immediate consequences. If the volatility σ is locally linear, then it vanishes.

Corollary 4.3. *Suppose there exist a linear operator* $S \in L(V, W)$ *and a non-empty open subset* $O \subset V$, *such that* $\sigma(h_0 + v) = Sv$, *for all* $v \in O$. *Then, we have* $S = 0$.

Proof. Setting $Y := \sigma(h_0 + O) = S(O) \subset \operatorname{ran} S$, we have $Y \subset \sigma(h_0 + V)$ and, by the open mapping theorem, the range Y is open in $\operatorname{ran} S$. Using theorem 4.1, it follows that $S = 0$. ■

Corollary 4.4 concerns the case of constant direction volatility.

Corollary 4.4. *If* $\dim W = 1$, *then* σ *is constant on* $h_0 + V$.

Proof. There exists $\lambda \in W$ with $W = \langle \lambda \rangle$. Suppose that σ is not constant on $h_0 + V$. Then, there exist $a, b \in \mathbb{R}$ with $a < b$ and $a\lambda, b\lambda \in \sigma(h_0 + V)$. The set $Y := \{\theta\lambda : \theta \in (a, b)\} \subset W$ is open in W, and by the continuity of σ we obtain $Y \subset \sigma(h_0 + V)$, which contradicts theorem 4.1. ■

Remark 4.5. The assumption $\dim W = 1$ implies that on $h_0 + V$, the volatility σ is of the form $\sigma(h) = \Phi(h)\lambda$ with a real-valued mapping Φ and a function $\lambda \in H^0_{\beta'}$, whence we speak about constant direction volatility. As we shall see in §7, in this particular situation, we can replace (4.1) by the weaker condition $F(\mathbb{R}) \neq 0$, and condition (4.2) can be skipped.

Our goal for the rest of this section is the proof of theorem 4.1. The first statement of theorem 4.1 immediately follows from lemma 2.6. According to Tappe (2010, lemma 4.3), the integral operator

$$T : H^0_{\beta'} \to H_\beta, \quad T\lambda := -\int_0^\bullet \lambda(\eta)\,\mathrm{d}\eta$$

is a bounded linear operator, and it is injective. We define the mapping $\Sigma := T \circ \sigma : H_\beta \to H_\beta$.

Lemma 4.6. *We have $V \subset \mathcal{D}(\mathrm{d}/\mathrm{d}x)$, and there exists $g_0 \in H_\beta$ such that*

$$\frac{\mathrm{d}}{\mathrm{d}x} v + \frac{\mathrm{d}}{\mathrm{d}x} \Psi(\Sigma(h_0 + v)) + g_0 \in V, \quad \textit{for all } v \in V. \tag{4.3}$$

Proof. We apply theorem 2.3 to the invariant foliations $(\mathcal{M}_t^{(0)})_{t\geq 0}$ and $(\mathcal{M}_t^{(h_0)})_{t\geq 0}$, implying $V \subset \mathcal{D}(\mathrm{d}/\mathrm{d}x)$ and the existence of some $h_0' \in H_\beta$ such that

$$\nu(h_0 + v) + h_0' \in V, \quad \text{for all } v \in V. \tag{4.4}$$

Inserting the HJM drift condition (1.3) into (4.4) gives us relation (4.3). ■

Now, the third statement of theorem 4.1 is a direct consequence of relation (4.3).

Remark 4.7. Integrating (4.3), we see that the linear space

$$V^\Psi := \langle \Psi(\Sigma(h_0 + v)) : v \in V \rangle \tag{4.5}$$

must necessarily be finite dimensional. In the present situation, by (3.1) and (3.2), the cumulant-generating function Ψ is no polynomial, and hence, this condition is difficult to ensure without σ being constant on $h_0 + V$.

Note that for the proof of theorem 4.1, we have not used conditions (4.1) and (4.2) up to this point. We shall now prove the second statement of theorem 4.1. In the following, for $z_0 \in \mathbb{R}^n$ and $\delta > 0$, we denote by $B_\delta(z_0)$ the open ball

$$B_\delta(z_0) := \{z \in \mathbb{R}^n : \|z - z_0\|_{\mathbb{R}^n} < \delta\}.$$

Proof (of the second statement of theorem 4.1). Suppose there is a subspace $U \subset W$ with $\dim U \geq 1$ and a set $Y \subset \sigma(h_0 + V)$ with $Y \cap U \neq \emptyset$ such that $Y \cap U$ is open in U. We will derive the contradiction

$$\dim V^\Psi = \infty. \tag{4.6}$$

In order to prove (4.6), by virtue of (3.1), (3.2) and (4.1), we may assume that $a_n \neq 0$, for all $n \in \mathbb{N}$. We set $E := T(U)$. Because T is injective, we have $\dim E = \dim U \geq 1$. By the open mapping theorem, the set $T(Y \cap U)$ is open in

E. Because $T(Y \cap U) \subset \Sigma(h_0 + V)$, there exist a direct sum decomposition $E = E_1 \oplus E_2$ with $\dim E_1 \geq 1$, elements $\Lambda_1 \in E_1$, $\Lambda_2 \in E_2$ with $\Lambda_1 \neq 0$, and constants $a, b \in \mathbb{R}$ with $a < b$ such that

$$\theta \Lambda_1 + \Lambda_2 \in \Sigma(h_0 + V), \quad \text{for all } \theta \in (a, b). \tag{4.7}$$

Now, let $m \in \mathbb{N}$ be arbitrary. By (4.7), there exist $\theta_1, \ldots, \theta_m \in (a, b)$ with $\theta_i \neq \theta_j$ for $i \neq j$ such that

$$\theta_i \Lambda_1 + \Lambda_2 \in \Sigma(h_0 + V), \quad \text{for all } i = 1, \ldots, m.$$

We will show that

$$\dim \langle \Psi(\theta_i \Lambda_1 + \Lambda_2) : i = 1, \ldots, m \rangle = m. \tag{4.8}$$

Indeed, let $\xi_1, \ldots, \xi_m \in \mathbb{R}$ be such that

$$\sum_{i=1}^{m} \xi_i \Psi(\theta_i \Lambda_1 + \Lambda_2) = 0. \tag{4.9}$$

By the power series representation (3.1), there exists $\eta > 0$ such that for all $(y, z) \in B_\eta(0)$, we obtain

$$\begin{aligned} \sum_{i=1}^{m} \xi_i \Psi(\theta_i y + z) &= \sum_{i=1}^{m} \xi_i \sum_{n=0}^{\infty} a_n (\theta_i y + z)^n \\ &= \sum_{n=0}^{\infty} a_n \sum_{i=1}^{m} \xi_i (\theta_i y + z)^n = \sum_{n=0}^{\infty} a_n \sum_{i=1}^{m} \xi_i \sum_{\substack{k,l \in \mathbb{N}_0 \\ k+l=n}} (\theta_i y)^k z^l \\ &= \sum_{n=0}^{\infty} a_n \sum_{\substack{k,l \in \mathbb{N}_0 \\ k+l=n}} \left(\sum_{i=1}^{m} \xi_i \theta_i^k \right) y^k z^l. \end{aligned}$$

Hence, defining the coefficients

$$c_{(k,l)} := a_{k+l} \sum_{i=1}^{m} \xi_i \theta_i^k, \quad (k, l) \in \mathbb{N}_0^2, \tag{4.10}$$

there is a bijection $\pi : \mathbb{N}_0 \to \mathbb{N}_0^2$ such that the power series

$$\sum_{\substack{n=0 \\ (k,l)=\pi(n)}}^{\infty} c_{(k,l)} y^k z^l \tag{4.11}$$

converges for all $(y, z) \in B_\eta(0)$. According to proposition A.3, there exists $r > 0$ such that the power series (4.11) converges absolutely and uniformly on $K_r(0)$—which denotes the compact ball defined in (A 1)—to a continuous function

$$f : K_r(0) \to \mathbb{R}, \quad f(y, z) = \sum_{(k,l) \in \mathbb{N}_0^2} c_{(k,l)} y^k z^l.$$

We claim that

$$c_{(k,0)} = 0, \quad \text{for all } k \in \mathbb{N}_0. \tag{4.12}$$

Indeed, suppose that (4.12) is not satisfied. Then, there exists $k_0 \in \mathbb{N}_0$ such that $c_{(k_0,0)} \neq 0$ and $c_{(k,0)} = 0$ for $k < k_0$. Because $a_n \neq 0$, for all $n \in \mathbb{N}_0$, by (4.10), for all $k < k_0$ and $l \in \mathbb{N}_0$, we obtain

$$c_{(k,l)} = a_{k+l} \sum_{i=1}^{m} \xi_i \theta_i^k = \frac{a_{k+l}}{a_k} c_{(k,0)} = 0.$$

Because the power series (4.11) converges absolutely for all $(y, z) \in B_\eta(0)$, we deduce that for some bijection $\tau : \mathbb{N}_0 \to \mathbb{N}_0^2$, the power series

$$\begin{aligned} \sum_{\substack{n=0 \\ (k,l)=\tau(n)}}^{\infty} c_{(k_0+k,l)} y^k z^l &= \frac{1}{y^{k_0}} \sum_{\substack{n=0 \\ (k,l)=\tau(n)}}^{\infty} c_{(k_0+k,l)} y^{k_0+k} z^l \\ &= \frac{1}{y^{k_0}} \sum_{\substack{n=0 \\ (k,l)=\pi(n)}}^{\infty} c_{(k,l)} y^k z^l \end{aligned} \tag{4.13}$$

also converges for all $(y, z) \in B_\eta(0)$ with $y \neq 0$. According to proposition A.3, the power series (4.13) converges absolutely and uniformly on $K_r(0)$ to a continuous function

$$g : K_r(0) \to \mathbb{R}, \quad g(y, z) = \sum_{(k,l) \in \mathbb{N}_0^2} c_{(k_0+k,l)} y^k z^l.$$

Moreover, for each $(y, z) \in K_r(0)$ with $y \neq 0$, we have

$$g(y, z) = \frac{f(y, z)}{y^{k_0}}.$$

Setting $\Lambda := (\Lambda_1, \Lambda_2)$ by (4.9), we have

$$f(y, z) = 0, \quad \text{for all } (y, z) \in \Lambda(\mathbb{R}_+) \cap K_r(0). \tag{4.14}$$

Because $\Lambda_1 \neq 0$, by (4.2), there exists a sequence $(x_n)_{n \in \mathbb{N}} \subset (0, \infty)$ with $x_n \to 0$ and $\Lambda_1(x_n) \neq 0$, for all $n \in \mathbb{N}$. Because Λ is continuous with $\Lambda(0) = (0, 0)$, setting $(y_n, z_n) := \Lambda(x_n)$, $n \in \mathbb{N}$, we have $(y_n, z_n) \to (0, 0)$. By (4.14), we obtain the contradiction

$$c_{(k_0,0)} = g(0, 0) = \lim_{n \to \infty} g(y_n, z_n) = \lim_{n \to \infty} \frac{f(y_n, z_n)}{y_n^{k_0}} = 0.$$

Consequently, we have (4.12). Because $a_n \neq 0$ for all $n \in \mathbb{N}$, by definition (4.10), we get

$$\sum_{i=1}^{m} \xi_i \theta_i^k = 0, \quad \text{for all } k \in \mathbb{N}_0.$$

It follows that $B\xi = 0$, where $B \in \mathbb{R}^{m \times m}$ denotes the Vandermonde matrix $B_{ki} = \theta_i^k$ for $k = 0, \ldots, m-1$ and $i = 1, \ldots, m$. Because $\theta_i \neq \theta_j$ for $i \neq j$, we deduce that $\xi_1 = \cdots = \xi_m = 0$, which proves (4.8). Because $m \in \mathbb{N}$ was arbitrary, we obtain (4.6), which contradicts remark 4.7. This completes the proof of theorem 4.1. ■

5. Sufficient conditions for the existence of affine realizations

In this section, we shall derive sufficient conditions for the existence of affine realizations for Lévy term structure models.

We suppose that the volatility $\sigma \in H^{\Psi}_{\beta,\beta'}$ is Lipschitz continuous and bounded. According to lemma 3.3, the drift α_{HJM} is Lipschitz continuous, too.

We have seen that for the existence of an affine realization, the linear spaces V^{Ψ} defined in (4.5) must necessarily be finite dimensional. As discussed in remark 4.7 (and shown in theorem 4.1), this is difficult to ensure with a driving Lévy process having jumps, unless the volatility σ is constant on the affine spaces generating the realization. Therefore, and because of theorem 4.1, we make the following assumptions:

— there exists a finite-dimensional subspace $W \subset H_\beta$ with $\sigma(H_\beta) \subset W$;
— each $w \in W$ is quasi-exponential, then, the linear space

$$V := \left\langle \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n w : w \in W \text{ and } n \in \mathbb{N}_0 \right\rangle$$

is finite dimensional; and
— for each $h_0 \in \mathcal{D}(\mathrm{d}/\mathrm{d}x)$, the volatility σ is constant on $h_0 + V$.

Theorem 5.1. *If the previous conditions are satisfied, then the HJMM equation (1.2) has a minimal realization generated by* V.

Proof. Let $h_0 \in \mathcal{D}(\mathrm{d}/\mathrm{d}x)$ be arbitrary. Because $V \subset \mathcal{D}(\mathrm{d}/\mathrm{d}x)$ and $\sigma(H_\beta) \subset W$, we have $\sigma(\mathcal{D}(\mathrm{d}/\mathrm{d}x)) \subset \mathcal{D}(\mathrm{d}/\mathrm{d}x)$ and σ is Lipschitz continuous and bounded on $(\mathcal{D}(\mathrm{d}/\mathrm{d}x), \|\cdot\|_{\mathcal{D}(\mathrm{d}/\mathrm{d}x)})$. By lemma 3.3, we have $\alpha_{\mathrm{HJM}}(\mathcal{D}(\mathrm{d}/\mathrm{d}x)) \subset \mathcal{D}(\mathrm{d}/\mathrm{d}x)$, and α_{HJM} is Lipschitz continuous on $(\mathcal{D}(\mathrm{d}/\mathrm{d}x), \|\cdot\|_{\mathcal{D}(\mathrm{d}/\mathrm{d}x)})$. Thus, according to Pazy (1983, theorem 6.1.7), there exists a classical solution $\psi \in C^1(\mathbb{R}_+; H_\beta)$ with $\psi(\mathbb{R}_+) \subset \mathcal{D}(\mathrm{d}/\mathrm{d}x)$ of the evolution equation

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\psi(t) = \dfrac{\mathrm{d}}{\mathrm{d}x}\psi(t) + \alpha_{\mathrm{HJM}}(\psi(t)), \\ \quad \psi(0) = h_0. \end{cases}$$

Defining the foliation $(\mathcal{M}_t^{(h_0)})_{t\geq 0}$ by $\mathcal{M}_t^{(h_0)} := \psi(t) + V$, relation (2.2) is fulfilled, and we have (2.4) because $\sigma(H_\beta) \subset W$. Let $t \geq 0$ and $v \in V$ be arbitrary. By the HJM drift condition (1.3), the drift α_{HJM} is constant on $\psi(t) + V$, and hence, we obtain

$$\begin{aligned} \nu(\psi(t) + v) &= \frac{\mathrm{d}}{\mathrm{d}x}\psi(t) + \frac{\mathrm{d}}{\mathrm{d}x}v + \alpha_{\mathrm{HJM}}(\psi(t) + v) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\psi(t) - \alpha_{\mathrm{HJM}}(\psi(t)) + \frac{\mathrm{d}}{\mathrm{d}x}v + \alpha_{\mathrm{HJM}}(\psi(t)) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\psi(t) + \frac{\mathrm{d}}{\mathrm{d}x}v \in \frac{\mathrm{d}}{\mathrm{d}t}\psi(t) + V = T\mathcal{M}_t, \end{aligned}$$

showing (2.3). Theorem 2.4 applies and yields that the foliation $(\mathcal{M}_t^{(h_0)})_{t\geq 0}$ is invariant for the HJMM equation (1.2). Consequently, the HJMM equation (1.2) has an affine realization generated by V. The minimality follows from theorem 4.1. ■

Remark 5.2. In particular, for every volatility structure of the form

$$\sigma(h) = \sum_{i=1}^{p} \Phi_i(h)\lambda_i, \quad h \in H_\beta,$$

with quasi-exponential functions $\lambda_1, \ldots, \lambda_p \in H_{\beta'}^0$, the HJMM equation (1.2) has a minimal realization generated by

$$V = \left\langle \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \lambda_1 : n \in \mathbb{N}_0 \right\rangle + \cdots + \left\langle \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \lambda_p : n \in \mathbb{N}_0 \right\rangle,$$

provided that for each $h_0 \in \mathcal{D}(\mathrm{d}/\mathrm{d}x)$, the mappings $\Phi_1, \ldots, \Phi_p : H_\beta \to \mathbb{R}$ are constant on the affine space $h_0 + V$.

6. Constant volatility

In this section, we apply our previous results for the particular case of a constant volatility $\sigma \in H_{\beta'}^0$ with $\sigma \neq 0$.

Corollary 6.1. *The following statements are equivalent*:

— *the HJMM equation* (*1.2*) *has an affine realization and*
— σ *is quasi-exponential.*

In either case, *the HJMM equation* (*1.2*) *has a minimal realization generated by* $V = \langle (\mathrm{d}/\mathrm{d}x)^n \sigma : n \in \mathbb{N}_0 \rangle$.

Proof. This is an immediate consequence of theorems 4.1 and 5.1. ■

Remark 6.2. Consequently, for constant volatility structures, we obtain exactly the same criterion for the existence of an affine realization as in the classical diffusion case, where the HJMM equation (1.2) is driven by a Wiener process, namely the function λ has to be quasi-exponential (Björk & Svensson 2001; Björk & Landén 2002; Tappe 2010).

7. Constant direction volatility

In this section, we shall tighten the statement of corollary 4.4 and present some consequences.

Throughout this section, we assume that the HJMM equation (1.2) has an affine realization generated by some subspace $V \subset H_\beta$. We suppose that the volatility $\sigma \in H_{\beta,\beta'}^{\Psi}$ is continuous. According to lemma 3.3, the drift α_{HJM} is continuous, too. Moreover, we assume that $\dim W = 1$, where $W := \langle \sigma(H_\beta) \rangle$, and that $F(\mathbb{R}) \neq 0$, where F denotes the Lévy measure of the driving Lévy process X in (1.2).

Theorem 7.1. *The following statements are true:*

— *for each $h_0 \in H_\beta$, the volatility σ is constant on the affine space $h_0 + V$ and*
— *each $v \in V$ is quasi-exponential, and we have $\langle (\mathrm{d}/\mathrm{d}x)^n v : n \in \mathbb{N}_0 \rangle \subset V$.*

Proof. Let $h_0 \in \mathcal{D}(\mathrm{d}/\mathrm{d}x)$ be arbitrary. Suppose that σ is not constant on $h_0 + V$. We will derive the contradiction

$$\dim V^{\Psi} = \infty, \tag{7.1}$$

where the linear space V^{Ψ} was defined in (4.5). Because T is injective, we have $\dim T(W) = 1$ with $T(W) = \langle \Sigma(H_\beta) \rangle$, and the mapping Σ is not constant on $h_0 + V$. There exists $\Lambda \in T(W)$ with $T(W) = \langle \Lambda \rangle$. Because Σ is not constant on $h_0 + V$, there exist $a, b \in \mathbb{R}$ with $a < b$ and $a\Lambda, b\Lambda \in \Sigma(h_0 + V)$. By the continuity of Σ, we obtain

$$\theta\Lambda \in \Sigma(h_0 + V), \quad \text{for all } \theta \in [a, b]. \tag{7.2}$$

Now, let $m \in \mathbb{N}$ be arbitrary. By (7.2), there exist $\theta_1, \ldots, \theta_m \in [a, b]$ with $|\theta_i| \neq |\theta_j|$ for $i \neq j$ such that

$$\theta_i \Lambda \in \Sigma(h_0 + V), \quad \text{for all } i = 1, \ldots, m.$$

We will show that

$$\dim \langle \Psi(\theta_i \Lambda) \rangle = m. \tag{7.3}$$

Indeed, let $\xi_1, \ldots, \xi_m \in \mathbb{R}$ be such that

$$\sum_{i=1}^{m} \xi_i \Psi(\theta_i \Lambda) = 0. \tag{7.4}$$

By the power series representation (3.1), there exists $\eta > 0$ such that

$$\begin{aligned} \sum_{i=1}^{m} \xi_i \Psi(\theta_i x) &= \sum_{i=1}^{m} \xi_i \sum_{n=0}^{\infty} a_n (\theta_i x)^n \\ &= \sum_{n=0}^{\infty} a_n \left(\sum_{i=1}^{m} \xi_i \theta_i^n \right) x^n, \quad x \in (-\eta, \eta), \end{aligned}$$

and we obtain

$$\sum_{n=0}^{\infty} a_n \left(\sum_{i=1}^{m} \xi_i \theta_i^n \right) x^n = 0, \quad x \in \Lambda(\mathbb{R}_+) \cap (-\eta, \eta).$$

Because $\lambda \neq 0$ and $\Lambda(0) = 0$, there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset \Lambda(\mathbb{R}_+) \cap (-\eta, \eta)$ with $x_n \neq 0$, $n \in \mathbb{N}$ and $x_n \to 0$. Therefore, the identity theorem for power series applies and yields

$$a_n \sum_{i=1}^{m} \xi_i \theta_i^n = 0, \quad n \in \mathbb{N}_0.$$

Because $F(\mathbb{R}) \neq 0$ by assumption, relations (3.1) and (3.2) show that $a_n > 0$ for every even $n \in \mathbb{N}$. It follows that $B\xi = 0$, where $B \in \mathbb{R}^{m\times m}$ denotes the Vandermonde matrix $B_{ki} = \theta_i^{2k}$ for $k = 1, \ldots, m$ and $i = 1, \ldots, m$. Because $|\theta_i| \neq |\theta_j|$ for $i \neq j$, we deduce that $\xi_1 = \cdots = \xi_m = 0$, which proves (7.3). Because $m \in \mathbb{N}$ was arbitrary, we conclude (7.1), which contradicts remark 4.7. Consequently, σ is constant on $h_0 + V$.

Now, let $h_0 \in H_\beta$ be arbitrary. Because $\mathcal{D}(\mathrm{d}/\mathrm{d}x)$ is dense in H_β, there exists a sequence $(h_n)_{n\in\mathbb{N}} \subset \mathcal{D}(\mathrm{d}/\mathrm{d}x)$ with $h_n \to h_0$. By the continuity of σ, for each $v \in V$, we obtain

$$\sigma(h_0) = \lim_{n\to\infty} \sigma(h_n) = \lim_{n\to\infty} \sigma(h_n + v) = \sigma(h_0 + v),$$

showing that σ is constant on $h_0 + V$. The second statement follows from relation (4.3). ■

Remark 7.2. The assumption dim $W = 1$ implies that the volatility σ is of the form $\sigma(h) = \Phi(h)\lambda$ with a real-valued mapping $\Phi : H_\beta \to \mathbb{R}$ and a function $\lambda \in H_{\beta'}^0$, whence we speak about constant direction volatility. Theorem 7.1 shows that in the presence of jumps, we obtain restrictions on the mapping Φ, which do occur in the Wiener driven case (Björk & Svensson 2001; Björk & Landén 2002; Tappe 2010).

Now, we assume that $\sigma = \phi \circ \ell$ with a continuous mapping $\phi : \mathbb{R} \to H_{\beta'}^0$ and a continuous linear functional $\ell : H_\beta \to \mathbb{R}$. We suppose that $\ell(W) = \mathbb{R}$.

Corollary 7.3. *The following statements are true*:

— *the volatility σ is constant and*
— *σ is quasi-exponential, and we have $\langle (\mathrm{d}/\mathrm{d}x)^n \sigma : n \in \mathbb{N}_0 \rangle \subset V$.*

Proof. Because $W \subset V$ by lemma 2.6, applying theorem 7.1 with $h_0 = 0$ yields that the volatility σ is constant on W. Note that $\ell|_W : W \to \mathbb{R}$ is an isomorphism. Therefore, for all $x, y \in \mathbb{R}$, we obtain

$$\phi(x) = \phi(\ell(\ell^{-1}x)) = \sigma(\ell^{-1}x) = \sigma(\ell^{-1}y) = \phi(\ell(\ell^{-1}y)) = \phi(y),$$

showing that σ is constant. The second statement follows from theorem 7.1. ■

Now, we assume that, in addition, dim $V = 1$. Then, according to lemma 2.6, we have $V = W$.

Corollary 7.4. *There are $\rho \in \mathbb{R}, \rho \neq 0$ and $\theta \in (\beta'/2, \infty)$ such that*

$$\sigma \equiv \rho\,\mathrm{e}^{-\theta\bullet}. \tag{7.5}$$

Proof. By corollary 7.3, the volatility σ is constant, and we have $\langle (\mathrm{d}/\mathrm{d}x)^n \sigma : n \in \mathbb{N}_0 \rangle \subset V$. Because $\dim V = 1$, we obtain that (7.5) is satisfied for some $\rho \in \mathbb{R}$, $\rho \neq 0$ and $\theta \in \mathbb{R}$. By definition (3.3) of the norm $\| \cdot \|_{\beta'}$, we have

$$\int_{\mathbb{R}_+} |\lambda'(x)|^2 \, \mathrm{e}^{\beta' x} \, \mathrm{d}x < \infty,$$

and, by definition (3.4) of the subspace $H^0_{\beta'}$, we have

$$\lim_{x \to \infty} \lambda(x) = 0.$$

We conclude that $\theta \in (\beta'/2, \infty)$, which finishes the proof. ■

From the literature (Jeffrey 1995; Björk & Svensson 2001; Filipović & Teichmann 2004), it is well known that for Wiener driven interest rate models, the following three types of affine short-rate realizations exist:

— the Ho–Lee model;
— the Hull–White extension of the Vasiček model; and
— the Hull–White extension of the Cox–Ingersoll–Ross model.

The evaluation at the short end $\ell : H_\beta \to \mathbb{R}$, $\ell(h) := h(0)$ is a continuous linear functional (Tappe 2010, theorem 4.1). Thus, applying corollary 7.4 for the Lévy case with jumps, we recognize (7.5) as the Hull–White extension of the Vasiček model, whereas an analogue for the Hull–White extension of the Cox–Ingersoll–Ross model does not exist.

Remark 7.5. The Ho–Lee model would correspond to (7.5) with $\theta = 0$. Note that in our framework, this volatility is even excluded in the Wiener case because of the technical reason that $\alpha_{\mathrm{HJM}} \notin H_\beta$. Indeed, for $\alpha_{\mathrm{HJM}} \in H_\beta$, one necessarily needs that $\lim_{x\to\infty} \sigma(x) = 0$, see relation (5.13) in Filipović (2001), which is not satisfied for $\sigma \equiv \rho$ with $\rho \neq 0$.

If the volatility σ is of the form (7.5), then the HJMM equation (1.2) has a one-dimensional realization, see corollary 6.1. By a well-known technique (Tappe 2010, proposition 2.8), we can choose the short rate $r_t(0)$ as a state process, whence we speak about a short-rate realization.

The author is grateful to Damir Filipović, Michael Kupper, Elisa Nicolato, Daniel Rost, David Skovmand, Josef Teichmann and Vilimir Yordanov for their helpful remarks and discussions. The author is also grateful to two anonymous referees for their helpful comments and suggestions.

Appendix A. Results about power series with several variables

For the proof of theorem 4.1, we require some results about power series with several variables. Because these results were not immediately available in the literature, we provide self-contained proofs in this appendix.

Lemma A.1. *Let* $(a_k)_{k\in\mathbb{N}_0} \subset \mathbb{R}$ *and* $(b_l)_{l\in\mathbb{N}_0} \subset \mathbb{R}$ *be sequences such that the series* $\sum_{k\in\mathbb{N}_0} a_k$ *and* $\sum_{l\in\mathbb{N}_0} b_l$ *are absolutely convergent. Then, the series*

$$\sum_{(k,l)\in\mathbb{N}_0^2} a_k b_l$$

is also absolutely convergent, and we have

$$\sum_{(k,l)\in\mathbb{N}_0^2} a_k b_l = \left(\sum_{k\in\mathbb{N}_0} a_k\right)\cdot\left(\sum_{l\in\mathbb{N}_0} b_l\right).$$

Proof. This is a direct consequence of the Cauchy product formula for absolutely convergent series (Forster 2011, Satz 8.3). ∎

In what follows, let $p \in \mathbb{N}$ be a positive integer. Let $K \subset \mathbb{R}^p$ be a compact subset. For a function $f : K \to \mathbb{R}$, we define the supremum norm

$$\|f\|_K := \sup\{|f(x)| : x \in K\}.$$

We will need the following version of Weierstrass' criterion of uniform convergence.

Lemma A.2. *Let* $f_n : K \to \mathbb{R}, n \in \mathbb{N}_0$ *be functions such that* $\sum_{n=0}^{\infty} \|f_n\|_K < \infty$. *Then, the series* $\sum_{n=0}^{\infty} f_n$ *converges absolutely and uniformly on* K *to a continuous function*

$$f : K \to \mathbb{R}, \quad f(z) = \sum_{n\in\mathbb{N}} f_n.$$

Proof. We can literally adapt the proof for functions with one variable (Forster 2011, Satz 21.2). ∎

For $x \in \mathbb{R}^p$ and $k \in \mathbb{N}_0^p$, we introduce the notation

$$x^k := x_1^{k_1} \cdot \ldots \cdot x_p^{k_p}.$$

For $a \in \mathbb{R}^p$ and $r > 0$, let $K_r(a)$ be the compact ball

$$K_r(a) := \{x \in \mathbb{R}^p : \|x - a\|_{\mathbb{R}^p} \le r\}. \tag{A 1}$$

Proposition A.3. *Let* $\pi : \mathbb{N}_0 \to \mathbb{N}_0^p$ *be a bijective mapping, let* $(c_n)_{n\in\mathbb{N}_0^p} \subset \mathbb{R}$ *and* $a \in \mathbb{R}^p$ *be such that the power series*

$$\sum_{\substack{n=0\\k=\pi(n)}}^{\infty} c_k (x-a)^k \tag{A 2}$$

converges for some $x \in \mathbb{R}^p$ *with* $x_i \neq a_i$, *for all* $i = 1, \ldots, p$. *Then, for all* $0 < r < \min\{|x_1 - a_1|, \ldots, |x_p - a_p|\}$, *the power series* (A 2) *converges absolutely and uniformly on* $K_r(a)$ *to a continuous function*

$$f : K_r(a) \to \mathbb{R}, \quad f(z) = \sum_{k\in\mathbb{N}_0^p} c_k (z-a)^k.$$

Proof. For each $k \in \mathbb{N}_0^p$, we define the function

$$f_k : \mathbb{R}^p \to \mathbb{R}, \quad f_k(z) := c_k(z-a)^k.$$

Because the series (A 2) converges, there exists a constant $M \geq 0$ such that

$$|f_k(x)| \leq M, \quad \text{for all } k \in \mathbb{N}_0^p.$$

Let $0 < r < \min\{|x_1 - a_1|, \ldots, |x_p - a_p|\}$ be arbitrary. We define the vector

$$\Theta := \left(\frac{r}{|x_1 - a_1|}, \ldots, \frac{r}{|x_p - a_p|} \right) \in (0,1)^p.$$

For all $z \in K_r(a)$ and $k \in \mathbb{N}_0^p$, we obtain

$$\begin{aligned}
|f_k(z)| = |c_k(z-a)^k| &= |c_k(x-a)^k| \frac{|(z-a)^k|}{|(x-a)^k|} \\
&= |f_k(x)| \frac{|z_1 - a_1|^{k_1} \cdot \ldots \cdot |z_p - a_p|^{k_p}}{|x_1 - a_1|^{k_1} \cdot \ldots \cdot |x_p - a_p|^{k_p}} \\
&\leq M \frac{r^{k_1} \cdot \ldots \cdot r^{k_p}}{|x_1 - a_1|^{k_1} \cdot \ldots \cdot |x_p - a_p|^{k_p}} = M\Theta_1^{k_1} \cdot \ldots \cdot \Theta_p^{k_p} = M\Theta^k.
\end{aligned}$$

By the geometric series and lemma A.1, the series

$$\sum_{k \in \mathbb{N}_0^p} \Theta^k = \prod_{i=1}^p \left(\sum_{k \in \mathbb{N}_0} \Theta_i^k \right)$$

converges absolutely. Therefore, we obtain

$$\sum_{\substack{n=0 \\ k=\pi(n)}}^{\infty} \|f_k\|_{K_r(a)} < \infty,$$

and hence, applying lemma A.2 concludes the proof. ■

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